NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2956

CREEP-BUCKLING ANALYSIS OF RECTANGULAR-SECTION COLUMNS

By Charles Libove

Langley Aeronautical Laboratory
Langley Field, Va.

Washington
June 1953

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SUMMARY

A previous analysis of the creep behavior of a slightly curved pin-ended H-section column under constant load is extended to the slightly curved solid rectangular-section column. The analysis leads to a differential equation for the plastic strains at the midheight cross section. The form of the equation indicates the significant parameters which may be useful in plotting test data on the creep life of columns. These are a lifetime parameter t'_{cr}, an initial-straightness parameter S or S', and the ratio of the average applied stress to the Euler stress $\overline{\sigma}/\sigma_E$. A numerical method of solving the differential equation, suitable for use with a high-speed digital computer, is described, and typical computed results are given. The existence of a finite lifetime, although not evident from the differential equation, is argued intuitively and confirmed by the numerical computations.

INTRODUCTION

The high temperatures that can develop in aircraft during supersonic flight make it important to consider the possible limitations due to creep on the useful service life of the structural components. In a previous paper (ref. 1), the creep of a slightly curved pin-ended idealized H-section column under a constant load and constant temperature was studied theoretically. The material of the column was characterized by an assumed creep law, for constant uniaxial compressive stress and constant temperature, of the form

$$\epsilon = \frac{\sigma}{E} + Ae^{B\sigma}t^K$$

where ϵ is the total compressive strain, σ is the constant compressive stress, t is the time after application of the stress, and E, A, B, and K are material constants. This form was selected because it applied to at least two alloys: the 75S-T6 aluminum alloy at 600° F (ref. 2) and a low-alloy steel at 800° F and, possibly, 1,100° F (ref. 3).

Shanley's engineering hypotheses of creep (ref. 4) were used in order to generalize the assumed constant-stress creep law to cover situations in which the stress varies with time, the condition encountered in the fibers of a column undergoing continuous lateral deflection because of creep.

In the present paper, the analysis of reference 1 is extended to the solid rectangular-section column. Except for the shape of the cross section, the assumptions of reference 1 are retained. In addition to those cited previously concerning material behavior, these assumptions are: (a) that the initial shape is half a sine wave, (b) that the load on the column is applied rapidly enough so that negligible creep occurs during the loading period (that is, elastic behavior on loading) but not so rapidly that dynamic effects have to be considered, (c) that the shape of the deforming column remains at all times sinusoidal, (d) that plane sections perpendicular to the (curved) center line of the column before loading remain plane and perpendicular to the center line after loading, and (e) that the lateral deflections are small compared to the length of the column.

The present analysis is very similar in its basic assumptions to that of Higgins (ref. 5) which also deals with the solid rectangular-section column; however, the two developments are entirely different, with the present development making evident the significant parameters.

SYMBOLS

A, B, E, K	material constants
b	thickness of column
I_1, I_2	certain integrals, across column thickness, of functions of μ at midheight
L	length of column
n	number of equal spaces into which cross-sectional thickness is assumed divided
P	load parameter, $\dfrac{1}{1 - \dfrac{\bar{\sigma}}{\sigma_E}}$
R_i	functions of the values of μ_i, defined by equation (22)

S	straightness parameter, $e^{-\frac{6B\bar{\sigma}}{K}\frac{\delta_0}{b}\frac{\sigma_E}{\sigma_E-\bar{\sigma}}}$
S'	alternate straightness parameter, $e^{-\frac{6B\bar{\sigma}}{K}\frac{\delta_0}{b}}$
t	time
t_{cr}	column lifetime, value of t at which lateral deflection become infinite
t'	time parameter, $t\left(\frac{AEB}{K}\right)^{1/K}e^{B\bar{\sigma}/K}$
t'_{cr}	lifetime parameter, $t_{cr}\left(\frac{AEB}{K}\right)^{1/K}e^{B\bar{\sigma}/K}$
w	width of column
z	distance measured from center line of column, positive toward convex side
$z_1, z_2, \ldots z_n$	coordinates z of centers of the n spaces (starting from concave side)
δ_0	amplitude of initial sinusoidal deflection
δ_1	midheight lateral deflection (not including δ_0)
ϵ	compressive strain
$\bar{\epsilon}$	average compressive strain at midheight of column
μ	plastic-strain parameter, $\frac{EB}{K}\left(\epsilon - \frac{\sigma}{E}\right)$
$\mu_1, \mu_2, \ldots \mu_n$	values of μ at centers of spaces
ξ	dummy coordinate representing z
σ	compressive stress
$\bar{\sigma}$	average compressive stress on column
σ_E	Euler buckling stress of column, $\frac{\pi^2 E}{12(L/b)^2}$

ANALYSIS

Assumed Law of Material Behavior

Because the longitudinal stress in any fiber of a continuously deflecting column varies with time, the description of the material given by the assumed constant-stress creep law mentioned in the introduction

$$\epsilon = \frac{\sigma}{E} + Ae^{B\sigma}t^{K} \tag{1}$$

is insufficient for purposes of analyzing the creep behavior of a column. Shanley's engineering hypotheses of creep (ref. 4) can be used to derive the following generalization of equation (1), which covers behavior under varying compressive stress: (See pp. 460-461 of ref. 1 for the derivation.)

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + f(\sigma,\epsilon) \tag{2}$$

where

$$f(\sigma,\epsilon) = \frac{K\left(Ae^{B\sigma}\right)^{1/K}}{\left(\epsilon - \frac{\sigma}{E}\right)^{\frac{1}{K}-1}} \tag{3}$$

and the dots denote differentiation with respect to time.

Equation (2) implies that the material behaves elastically under an infinitely rapid increase or reduction in stress. If equation (2) is rewritten with the term $\dot{\sigma}/E$ transposed to the left-hand side, $f(\sigma,\epsilon)$ is seen to represent the creep rate, or rate of growth, of the plastic strain $\epsilon - \frac{\sigma}{E}$.

Equation (3) implies that, if after a period of creep under compressive stress the stress is reduced to tension (σ negative), the creep rate will still remain positive - that is, compressive - although it may become exceedingly small as the stress takes on larger tensile values. The neglect of tensile creep will introduce some unconservatism into the analysis when the lateral deflection gets so large that the stress becomes tensile on the convex side of the column. The degree of unconservatism, however, should be small in many practical situations

in which the initial curvature is reasonably small and the average applied stress is reasonably high but not close to the Euler stress. In such cases the time between the occurrence of appreciable tension and the calculated ultimate collapse of the column will be a small part of the total column lifetime.

Qualitative Description of Column Behavior

The column, before application of load, is shown in figure 1(a). The cross section is rectangular, of thickness b and width w. A slight initial curvature is present in the form of half a sine wave of amplitude δ_0.

A load, less than the Euler load, is instantaneously applied and produces an average stress $\overline{\sigma}$ and an instantaneous static deflection $\delta_1(0)$, as shown in figure 1(b). By hypothesis, the material behaves elastically during instantaneous stressing; therefore, the conditions existing immediately after load application $(t = 0)$ are obtainable from an elastic analysis. At that instant no plastic strain exists anywhere in the column.

With the load held constant, the column will continue to deform because of creep and will deflect laterally because of larger creep strains on the more highly stressed, or concave, side than on the convex side (fig. 1(c)). The problem is to determine the history of the deformations.

That the present analysis should predict a finite lifetime for nonvanishing initial curvature can be established by the following intuitive argument: A solid rectangular-section column is certainly less stiff than the H-section column formed from it by concentrating half of the material of the column at each extreme fiber, all other conditions remaining unchanged. According to the analysis of reference 1, however, which is based on the same assumptions as the present analysis, the lateral deflection of any slightly curved H-section column approaches infinity in a finite time; hence, the deflection of the less stiff solid-section column must certainly become infinite in finite time.

Development of Basic Equations

For a rigorous analysis, equilibrium, compatibility, and the law of material behavior must be satisfied at all sections of the column at every instant. For simplicity, however, the deflection shape is assumed to be merely a magnification of the initial shape - that is, half a sine wave. Because of this reduction to a single degree of freedom in the

deflection shape, the requirements of equilibrium, and so forth, can at best be satisfied only in some average way over the length of the column. An even simpler solution, that followed herein, is to satisfy the requirements at only one section of the column, for example, at the midheight.

Figure 2 shows the upper half of the column of figure 1(c) as a free body. The compressive stress σ at the cut section varies from fiber to fiber and also varies in time - that is, $\sigma = \sigma(z,t)$. Similarly, the compressive strain $\epsilon = \epsilon(z,t)$.

Equilibrium of vertical forces requires that

$$\int_{-b/2}^{b/2} \sigma \, dz = \bar{\sigma} b \tag{4}$$

Equilibrium of moments requires that

$$\int_{-b/2}^{b/2} \sigma z \, dz = -\bar{\sigma} b(\delta_0 + \delta_1) \tag{5}$$

The assumed law of material behavior (eqs. (2) and (3)) gives the following equation for the strain rate:

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{K\left(Ae^{B\sigma}\right)^{1/K}}{\left(\epsilon - \frac{\sigma}{E}\right)^{\frac{1}{K} - 1}} \tag{6}$$

Finally, the assumptions that the shape is sinusoidal, that plane sections remain plane and perpendicular to the center line of the column, and that the deflections are small result in the following relationship between the strains and the lateral deflection at the column midheight:

$$\epsilon = -z \frac{\pi^2 \delta_1}{L^2} + \bar{\epsilon}$$

$$= -z \frac{12 \sigma_E \delta_1}{E b^2} + \bar{\epsilon} \tag{7}$$

where $\bar{\epsilon}$ is the average strain across the section, or

$$\bar{\epsilon} = (\epsilon)_{z=0} \tag{8}$$

and σ_E is the Euler buckling stress of the column; that is,

$$\sigma_E = \frac{\pi^2 E}{12(L/b)^2} \tag{9}$$

Reduction of Basic Equations

Equations (4) to (7) can be reduced to a single equation in terms of the elastic strain $\epsilon - \frac{\sigma}{E}$ at the midheight of the column, which is also a function of position and time. Equations (4), (5), and (6) are first rewritten as follows (with cognizance being taken of eq. (7)):

$$\begin{aligned}
\int_{-b/2}^{b/2} \left(\epsilon - \frac{\sigma}{E}\right) dz &= -\frac{\bar{\sigma} b}{E} + \int_{-b/2}^{b/2} \epsilon \, dz \\
&= -\frac{\bar{\sigma} b}{E} + \int_{-b/2}^{b/2} \left(\bar{\epsilon} - z\,\frac{12\sigma_E \delta_1}{Eb^2}\right) dz \\
&= b\left(\bar{\epsilon} - \frac{\bar{\sigma}}{E}\right)
\end{aligned} \tag{10}$$

$$\begin{aligned}
\int_{-b/2}^{b/2} \left(\epsilon - \frac{\sigma}{E}\right) z \, dz &= \frac{\bar{\sigma} b \left(\delta_0 + \delta_1\right)}{E} + \int_{-b/2}^{b/2} \epsilon z \, dz \\
&= \frac{\bar{\sigma} b \left(\delta_0 + \delta_1\right)}{E} + \int_{-b/2}^{b/2} \left(\bar{\epsilon} - z\,\frac{12\sigma_E \delta_1}{Eb^2}\right) z \, dz \\
&= \frac{\bar{\sigma} b \delta_0}{E} - \frac{\left(\sigma_E - \bar{\sigma}\right) b \delta_1}{E}
\end{aligned} \tag{11}$$

$$\frac{\partial}{\partial t}\left(\epsilon - \frac{\sigma}{E}\right) = \frac{KA^{1/K}e^{B\sigma/K}}{\left(\epsilon - \frac{\sigma}{E}\right)^{\frac{1}{K}-1}}$$

$$= \frac{KA^{1/K}e^{\frac{BE}{K}\left[-\left(\epsilon - \frac{\sigma}{E}\right)+\epsilon\right]}}{\left(\epsilon - \frac{\sigma}{E}\right)^{\frac{1}{K}-1}}$$

$$= \frac{KA^{1/K}e^{\frac{BE}{K}\left[-\left(\epsilon - \frac{\sigma}{E}\right)+\bar{\epsilon}-z\frac{12\sigma_E\delta_1}{Eb^2}\right]}}{\left(\epsilon - \frac{\sigma}{E}\right)^{\frac{1}{K}-1}} \tag{12}$$

Equation (12) can be reduced further if $\bar{\epsilon}$ is eliminated through use of equation (10) and δ_1 eliminated through use of equation (11). The result, after slight simplification, is the following equation in which $\epsilon - \frac{\sigma}{E}$ is the only unknown:

$$\frac{\partial}{\partial t}\left(\epsilon - \frac{\sigma}{E}\right)^{1/K} = A^{1/K} \exp \frac{BE}{K}\left\{-\left(\epsilon - \frac{\sigma}{E}\right) + \frac{1}{b}\int_{-b/2}^{b/2}\left(\epsilon - \frac{\sigma}{E}\right)d\xi + \frac{\bar{\sigma}}{E} - \right.$$

$$\left.\frac{z}{b}\left[\frac{\delta_0}{b}\,\frac{12\sigma_E\bar{\sigma}}{E\left(\sigma_E - \bar{\sigma}\right)} - \frac{12\sigma_E}{b^2\left(\sigma_E - \bar{\sigma}\right)}\int_{-b/2}^{b/2}\left(\epsilon - \frac{\sigma}{E}\right)\xi\, d\xi\right]\right\} \tag{13}$$

where ξ is a dummy variable representing z. Introducing the following parameters:

the plastic-strain parameter

$$\frac{EB}{K}\left(\epsilon - \frac{\sigma}{E}\right) \equiv \mu \tag{14}$$

the time parameter

$$t\left(\frac{AEB}{K}\right)^{1/K} e^{B\overline{\sigma}/K} \equiv t' \tag{15}$$

the initial-straightness parameter

$$e^{-\frac{6B\overline{\sigma}}{K} \frac{\delta_0}{b} \frac{\sigma_E}{\sigma_E - \overline{\sigma}}} \equiv S \tag{16}$$

and the load parameter

$$\frac{1}{1 - \frac{\overline{\sigma}}{\sigma_E}} \equiv P \tag{17}$$

permits equation (13) to be simplified to

$$\frac{\partial}{\partial t'} \mu^{1/K} = S^{2z/b} e^{\left(-\mu + \int_{-1/2}^{1/2} \mu \, d\frac{\xi}{b} + \frac{12zP}{b} \int_{-1/2}^{1/2} \mu \frac{\xi}{b} \, d\frac{\xi}{b}\right)} \tag{18}$$

where $\mu \equiv \mu(z,t)$ is subject to the initial condition $\mu(z,0) = 0$.

Equation (18) gives, essentially, the rate of growth of the plastic strain to the $1/K$ power for any fiber of the column as a function of the position z of that fiber, the instantaneous value of the plastic-strain parameter μ in that fiber, and the two integrals involving the instantaneous values of μ in all the fibers of the column.

For the initially perfectly straight column ($S = 1$), equation (18) will be satisfied by $\mu = t'^K$, as is expected, inasmuch as $\mu = t'^K$ is equivalent to equation (1), the law of material behavior for direct compression.

Numerical Method of Solution of Equation (18)

Equation (18) lends itself readily to numerical solution by means of the modified Euler method (ref. 6). The following adaptation of this method has been found suitable for use in conjunction with a high-speed digital computer, the National Bureau of Standards Eastern Automatic Computer (SEAC):

Let the cross section of the column at the midheight be divided into n equal spaces. The width of each space will be b/n and the centers of the spaces, starting at the left-hand, or concave, side of the column and going toward the right, will be located at $z = z_1 = -\frac{b}{2} + \frac{1}{2}\frac{b}{n}$, $z = z_2 = -\frac{b}{2} + \frac{3}{2}\frac{b}{n}$, ... Let the values of the plastic-strain parameter μ at the centers of these spaces be denoted by $\mu_1, \mu_2, \cdot \cdot \cdot \mu_n$, respectively. The two integrals appearing in equation (18) can then be approximated in terms of the parameters μ_i and z_i, the very simplest approximation for machine-calculation purposes being that obtained by assuming that the integrand is constant over the width of a space and equal to its value at the center of the space. Then,

$$\int_{-1/2}^{1/2} \mu \, d\frac{\xi}{b} \equiv I_1 \approx \frac{1}{n}\left(\mu_1 + \mu_2 + \cdot \cdot \cdot + \mu_n\right) \tag{19}$$

and

$$\int_{-1/2}^{1/2} \mu \frac{\xi}{b} \, d\frac{\xi}{b} \equiv I_2 \approx \frac{1}{n}\left[\mu_1\left(-1 + \frac{1}{n}\right) + \mu_2\left(-1 + \frac{3}{n}\right) + \cdot \cdot \cdot + \mu_n\left(-1 + \frac{2n - 1}{n}\right)\right] \tag{20}$$

Writing equation (18) for the specific $z = z_1, z_2, \ldots z_n$ gives the following system of ordinary differential equations which is approximately equivalent to equation (18):

$$\left.\begin{aligned}
\frac{d\left(\mu_1^{1/K}\right)}{dt'} &= S^{-1+\frac{1}{n}}e^{\left[-\mu_1+I_1+6\left(-1+\frac{1}{n}\right)PI_2\right]} \\
\frac{d\left(\mu_2^{1/K}\right)}{dt'} &= S^{-1+\frac{3}{n}}e^{\left[-\mu_2+I_1+6\left(-1+\frac{3}{n}\right)PI_2\right]} \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\frac{d\left(\mu_n^{1/K}\right)}{dt'} &= S^{-1+\frac{2n-1}{n}}e^{\left[-\mu_n+I_1+6\left(-1+\frac{2n-1}{n}\right)PI_2\right]}
\end{aligned}\right\} \tag{21}$$

Letting

$$R_i \equiv S^{-1+\frac{2i-1}{n}}e^{\left[-\mu_i+I_1+6\left(-1+\frac{2i-1}{n}\right)PI_2\right]} \tag{22}$$

simplifies equations (21) to:

$$\left.\begin{aligned}
\frac{d\left(\mu_1^{1/K}\right)}{dt'} &= R_1 \\
\frac{d\left(\mu_2^{1/K}\right)}{dt'} &= R_2 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\frac{d\left(\mu_n^{1/K}\right)}{dt'} &= R_n
\end{aligned}\right\} \tag{23}$$

The parameters R_i are functions of t' because they involve the $\mu_i(t')$.

Equations (23) define curves of the μ_i plotted against t' and can be solved in a step-by-step fashion subject to the initial conditions $\mu_i(0) = 0$. As may be expected on the basis of the physical argument presented previously, at least one of the μ_i, probably μ_1, will tend to approach infinity asymptotically at some finite value of t'. A step-by-step solution of equations (23) on the basis of equal increments of t' therefore should not be attempted. Rather, the solutions should be carried forward on the basis of equal increments in μ_1 or $\mu_1{}^{1/K}$, and the increment in t' should be regarded as one of the unknowns in each step. A detailed account of the calculation procedure follows.

Suppose that the curves of $\mu_1, \mu_2, \cdot \cdot \cdot \mu_n$ plotted against t' have been traced up to some particular value of t'. It is now desired to carry these curves forward one more step corresponding to a specified finite increment in μ_1 denoted by $\Delta\mu_1$. To be determined are the increment $\Delta t'$ in t' and the increments $\Delta\mu_2, \Delta\mu_3, \cdot \cdot \cdot \Delta\mu_n$ in the other μ_i corresponding to the specified increment $\Delta\mu_1$ in μ_1. Since the values μ_1 and $\Delta\mu_1$ are known, the increment $\Delta\left(\mu_1{}^{1/K}\right)$ in the quantity $\mu_1{}^{1/K}$ can be computed. The first of equations (23) then gives the following approximation to $\Delta t'$:

$$\Delta t' = \Delta\left(\mu_1{}^{1/K}\right)\frac{1}{R_1(t')} \tag{24}$$

With $\Delta t'$ known, the remaining equations give the increments $\Delta\left(\mu_2{}^{1/K}\right)$, $\Delta\left(\mu_3{}^{1/K}\right), \cdot \cdot \cdot \Delta\left(\mu_n{}^{1/K}\right)$ as follows:

$$\left.\begin{aligned}
\Delta\left(\mu_2{}^{1/K}\right) &= \Delta t' R_2(t') \\
\Delta\left(\mu_3{}^{1/K}\right) &= \Delta t' R_3(t') \\
\cdot \quad \cdot \quad \cdot \quad &\cdot \quad \cdot \\
\Delta\left(\mu_n{}^{1/K}\right) &= \Delta t' R_n(t')
\end{aligned}\right\} \tag{25}$$

From these increments, the values of $\mu_2, \mu_3, \ldots \mu_n$ corresponding to $t' + \Delta t'$ can be computed. The curves have now been tentatively advanced one step and tentative values of $R_i(t' + \Delta t')$ can be determined. Improved values of $\Delta t'$ and $\Delta\left(\mu_2{}^{1/K}\right), \Delta\left(\mu_3{}^{1/K}\right), \ldots \Delta\left(\mu_n{}^{1/K}\right)$ can now be obtained by replacing the right-hand sides of equations (24) and (25) by averages based on the values of R_i at the end of the interval as well as at the beginning of the interval, that is, by using the equations

$$\Delta t' = \Delta\left(\mu_1{}^{1/K}\right)\frac{1}{2}\left[\frac{1}{R_1(t')} + \frac{1}{R_1(t' + \Delta t')}\right] \tag{26}$$

$$\left.\begin{aligned}
\Delta\left(\mu_2{}^{1/K}\right) &= \Delta t' \ \frac{1}{2}\left[R_2(t') + R_2(t' + \Delta t')\right] \\
\Delta\left(\mu_3{}^{1/K}\right) &= \Delta t' \ \frac{1}{2}\left[R_3(t') + R_3(t' + \Delta t')\right] \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\Delta\left(\mu_n{}^{1/K}\right) &= \Delta t' \ \frac{1}{2}\left[R_n(t') + R_n(t' + \Delta t')\right]
\end{aligned}\right\} \tag{27}$$

The improved values of $\mu_i(t' + \Delta t')$ obtained from equations (27) give new values of $R_i(t' + \Delta t')$ which can be substituted in equations (26)

and (27) in order to obtain still better values of $\Delta t'$ and $\Delta\left(\mu_2^{1/K}\right)$, $\Delta\left(\mu_3^{1/K}\right)$, . . . $\Delta\left(\mu_n^{1/K}\right)$. This process can be repeated until no change is evident in the values of $\Delta t'$ and $\mu_2(t' + \Delta t')$, $\mu_3(t' + \Delta t')$, . . . $\mu_n(t' + \Delta t')$.

The procedure just described can also be used without modification to start the curves. It is necessary to use only the fact that $\mu_1(0) = 0$.

Approximate Solution of Equation (18) Based on $n = 2$

If, for purposes of numerical integration, the cross section were divided into two equal spaces only, then the system of equations (21) would reduce to the following pair of equations:

$$\left.\begin{aligned} \frac{d\left(\mu_1^{1/K}\right)}{dt'} &= S^{-1/2}e^{\left(\mu_1-\mu_2\right)\left(\frac{3P}{4}-\frac{1}{2}\right)} \\ \\ \frac{d\left(\mu_2^{1/K}\right)}{dt'} &= S^{1/2}e^{-\left(\mu_1-\mu_2\right)\left(\frac{3P}{4}-\frac{1}{2}\right)} \end{aligned}\right\} \tag{28}$$

With the differentiations indicated on the left-hand sides carried out, these equations become identical in form with the equations which arise in the analysis of the two-flange column - namely, equations (21) and (22) of reference 1. The solution presented in reference 1 can therefore be used to obtain the approximate solution of the present problem corresponding to $n = 2$. In order to use the solution in reference 1, certain substitutions should be made in the symbols and graph labels of that paper and, then, the results of that paper will apply directly to the present problem. The substitutions are as follows:

(1) Replace γ_R/γ_L by S

(2) Replace η_L by $\mu_1\left(\frac{3P}{4} - \frac{1}{2}\right)$

(3) Replace η_R by $\mu_2\left(\frac{3P}{4} - \frac{1}{2}\right)$

(4) Replace $\gamma_L t$ by $\left(\frac{3P}{4} - \frac{1}{2}\right)^{1/K} S^{-1/2} t'$

(5) Replace $\gamma_L t_{cr}$ by $\left(\frac{3P}{4} - \frac{1}{2}\right)^{1/K} S^{-1/2} t'_{cr}$

(6) The constant K is unchanged

Lateral Deflections, Strains, and Stresses

With the $\mu_i(t')$ completely determined by the solution of equation (18), the time history of the plastic strains $\epsilon - \frac{\sigma}{E}$ is, in effect, known. Equation (11) can then be used to compute the midheight lateral deflections $\delta_1(t)$, and equation (10) gives the history of the average midheight strain $\bar{\epsilon}(t)$. The strains $\epsilon(z,t)$ are then obtainable from equation (7). A knowledge of $\epsilon - \frac{\sigma}{E}$ and ϵ is sufficient to determine the midheight stresses $\sigma(z,t)$.

RESULTS AND DISCUSSION

Significant Parameters

The usefulness of data obtained from tests is enhanced if the data are presented in terms of the appropriate parameters. Equation (18) of the present analysis suggests that test data on the creep life of columns whose material creep curves are adequately represented by equation (1) may profitably be plotted in terms of the parameters K, t'_{cr}, S, and $\bar{\sigma}/\sigma_E$, where t'_{cr} is the value of t' corresponding to collapse of the column. If the analysis were perfectly applicable, the test data would give a single curve of t'_{cr} plotted against S for a given value of $\bar{\sigma}/\sigma_E$ and a given value of K. Scatter of test points away from such a curve will, of course, occur, because of shortcomings of the theory - for example, the neglect of third-stage creep in equation (1), the approximate engineering hypotheses used to generalize equation (1), the neglect of tensile creep in the generalization, the assumption of an exactly sinusoidal shape for both the initial and the subsequent deflections, and the assumption of instantaneous elastic loading.

Inasmuch as the parameter S contains the parameter $\bar{\sigma}/\sigma_E$ and each plotted curve or test point has a definite value of $\bar{\sigma}/\sigma_E$ associated with it, a simpler parameter S' may be used instead of S, where

$$S' = e^{-\frac{6B\sigma}{K}\frac{\delta_0}{b}}$$

$$= S^{1-\frac{\bar{\sigma}}{\sigma_E}} \tag{29}$$

If the test results for a single material at a single temperature are to be plotted, then A, B, E, and K are the same for all the tests and the significant parameters may be simplified further to $t_{cr}e^{B\bar{\sigma}/K}$, $\sigma\frac{\delta_0}{b}$, and $\bar{\sigma}/\sigma_E$.

Calculated Values of t'_{cr}

In the section entitled "Analysis" a numerical method of solving equation (18) was given, based on dividing the column cross section at the midheight into n equal spaces and writing equation (18) for the center of each space. This method has been carried out and found to be satisfactory for the case $K = 1$, $\bar{\sigma}/\sigma_E = 0.7$ and 0.9, and for several values of S. The value $n = 14$ was found to lead to sufficiently accurate results in a trial calculation, and this value was used for all the cases.

Figure 3 shows a typical curve of μ_1 plotted against t'. (The curves of μ_2, μ_3, . . . μ_{14} are not shown, but they would lie successively below the curve of μ_1.) This figure tends to confirm the intuitive argument presented previously to the effect that the deflections should approach infinity in a finite time. Computation was stopped when the computed increment in t' corresponding to the specified increment in μ_1 became less than the precision of the computation. The last computed value of t' was taken as being equal to the asymptotic value t'_{cr} for all practical purposes.

The values of t'_{cr} taken from a number of curves like the one in figure 3 are summarized in figure 4 in plots of t'_{cr} against S for $\bar{\sigma}/\sigma_e = 0.7$ and 0.9. The circles are the actually computed data through which the curves are faired.

As is to be expected, t'_{cr} tends toward zero for columns that are more and more initially curved ($S \longrightarrow 0$) and toward infinity for columns approaching straightness ($S \longrightarrow 1$). In order to show more clearly the

behavior of the curves in these two regions, the data of figure 4 are replotted in figures 5 and 6 in which logarithmic scales are used to magnify the region of $S = 0$ and $S = 1$, respectively.

All the computations were performed on the National Bureau of Standards Eastern Automatic Computer. The computation of a value of t'_{cr} took about 15 minutes, exclusive of the exploratory computation time required to establish optimum values of precision, the step size $\Delta\mu_1$, and the number of cross-sectional spaces n. The calculated values of t'_{cr} are felt to be accurate to within 2 percent.

An approximate solution for the solid rectangular-section column corresponding to $n = 2$, as is indicated in the section entitled "Analysis," already exists implicitly in the two-flange-column analysis in reference 1. Figure 5 of reference 1, which summarizes the theoretical lifetime data for two-flange columns, can therefore be used to yield approximate information on the lifetimes of solid columns by replacing the label γ_R/γ_L by the label S and the label $\gamma_L t_{cr}$ by the label

$$\left(\frac{3P}{4} - \frac{1}{2}\right)^{1/K} S^{-1/2} t'_{cr}$$

In figure 7, the approximate results thus obtained for the rectangular-section column for the cases where $K = 1$ and $\overline{\sigma}/\sigma_E = 0.7$ and 0.9 are compared with the much more accurate results for these cases based on computations with $n = 14$ and obtained from figure 4 of the present paper.

CONCLUDING REMARKS

On the basis of certain assumptions regarding the shape of a column and the creep behavior of the material under varying stress, an analysis has been made of a slightly curved solid rectangular-section pin-ended column carrying a constant load. The analysis led to a differential equation for the plastic strains at the midheight cross section as a function of time. The form of the equation indicates the significant parameters of the problem, which parameters may be useful in plotting test data on the creep life of columns. These are a lifetime parameter t'_{cr}, an initial-straightness parameter S or S', and the ratio of the average applied stress to the Euler stress $\overline{\sigma}/\sigma_E$.

A numerical method of solution of the differential equation, suitable for use with a high-speed digital computer, has been described, and typical computed results have been presented.

The existence of a finite lifetime, although not evident from the differential equation, was argued intuitively and confirmed by the numerical computations.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., March 31, 1953.

REFERENCES

1. Libove, Charles: Creep Buckling of Columns. Jour. Aero. Sci., vol. 19, no. 7, July 1952, pp. 459-467.

2. Jackson, L. R., Schwope, A. D., and Shober, F. R.: Stress-Strain-Time Properties of Some Aircraft Materials. Pt. III, ch. 17 of Weight-Strength Analysis of Aircraft Structures by F. R. Shanley. McGraw-Hill Book Co., Inc., 1952, pp. 307-322.

3. Fisher, J. C., and Holloman, J. H.: Wanted: Experimental Support for Theories of Plastic Flow. A Symposium on the Plastic Deformation of Crystalline Solids. Mellon Inst. (Pittsburgh), May 19-20, 1950, pp. 199-208. (Sponsored by Carnegie Inst. Tech. and ONR, Dept. Navy.)

4. Shanley, F. R.: Weight-Strength Analysis of Aircraft Structures. McGraw-Hill Book Co., Inc., 1952. Pt. III, ch. 16, pp. 265-305.

5. Higgins, T. P., Jr.: Effect of Creep on Column Deflection. Pt. III, ch. 20 of Weight-Strength Analysis of Aircraft Structures by F. R. Shanley. McGraw-Hill Book Co., Inc., 1952, pp. 359-385.

6. Scarborough, James B.: Numerical Mathematical Analysis. Second ed., The Johns Hopkins Press (Baltimore), 1950, pp. 235-240.

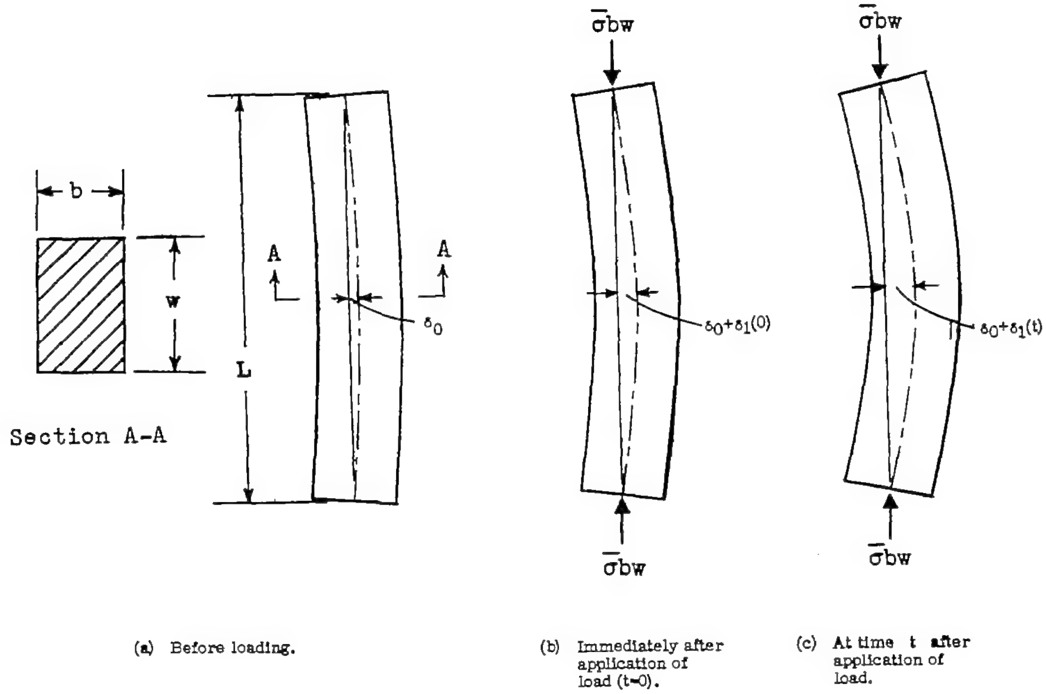

Figure 1.- Schematic description of column.

Figure 2.- Upper half of column of figure 1(c).

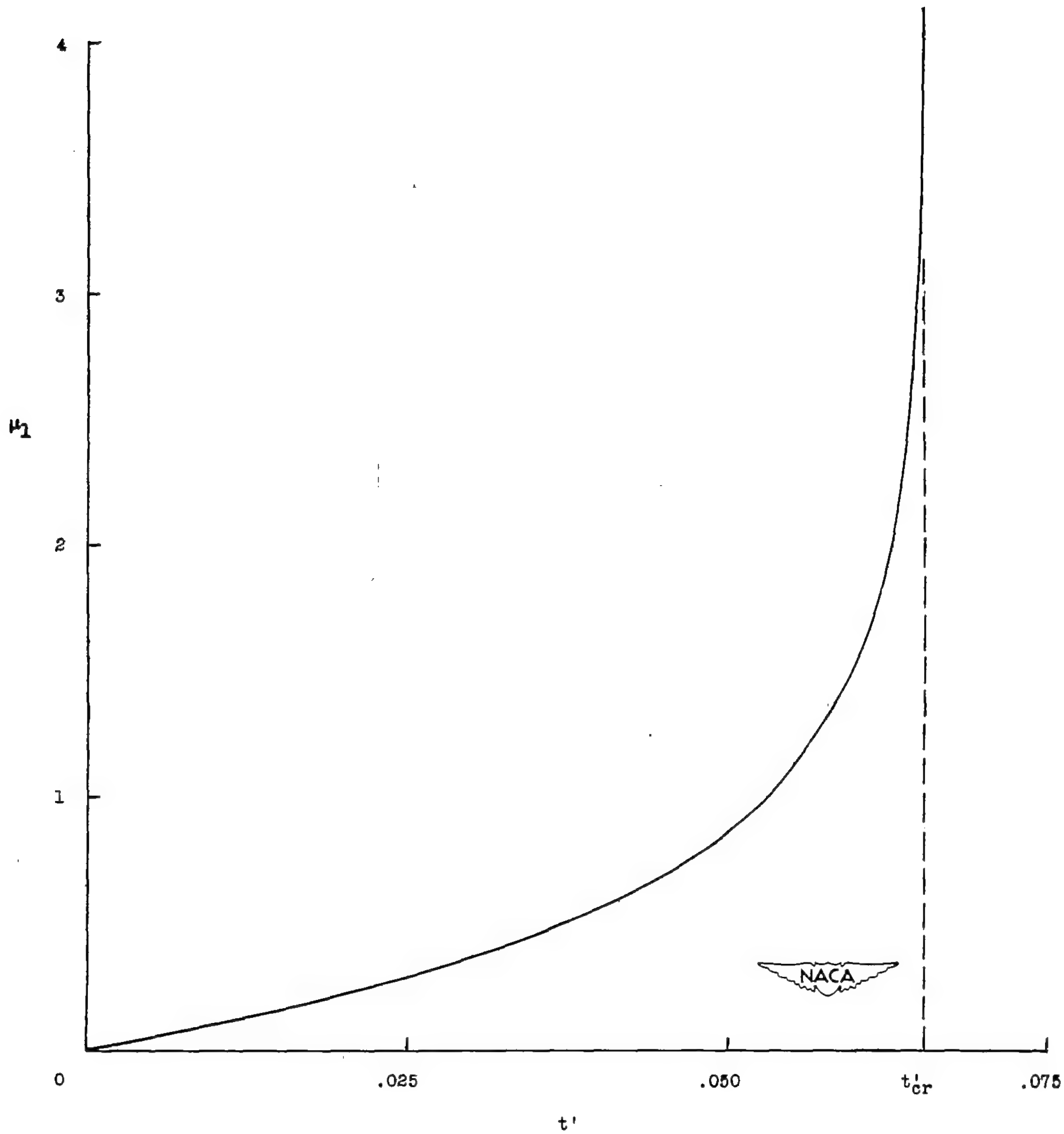

Figure 3.- Variation of plastic-strain parameter μ_1 with time parameter t'. $K = 1$; $\bar{\sigma}/\sigma_E = 0.7$; $S = 0.1$.

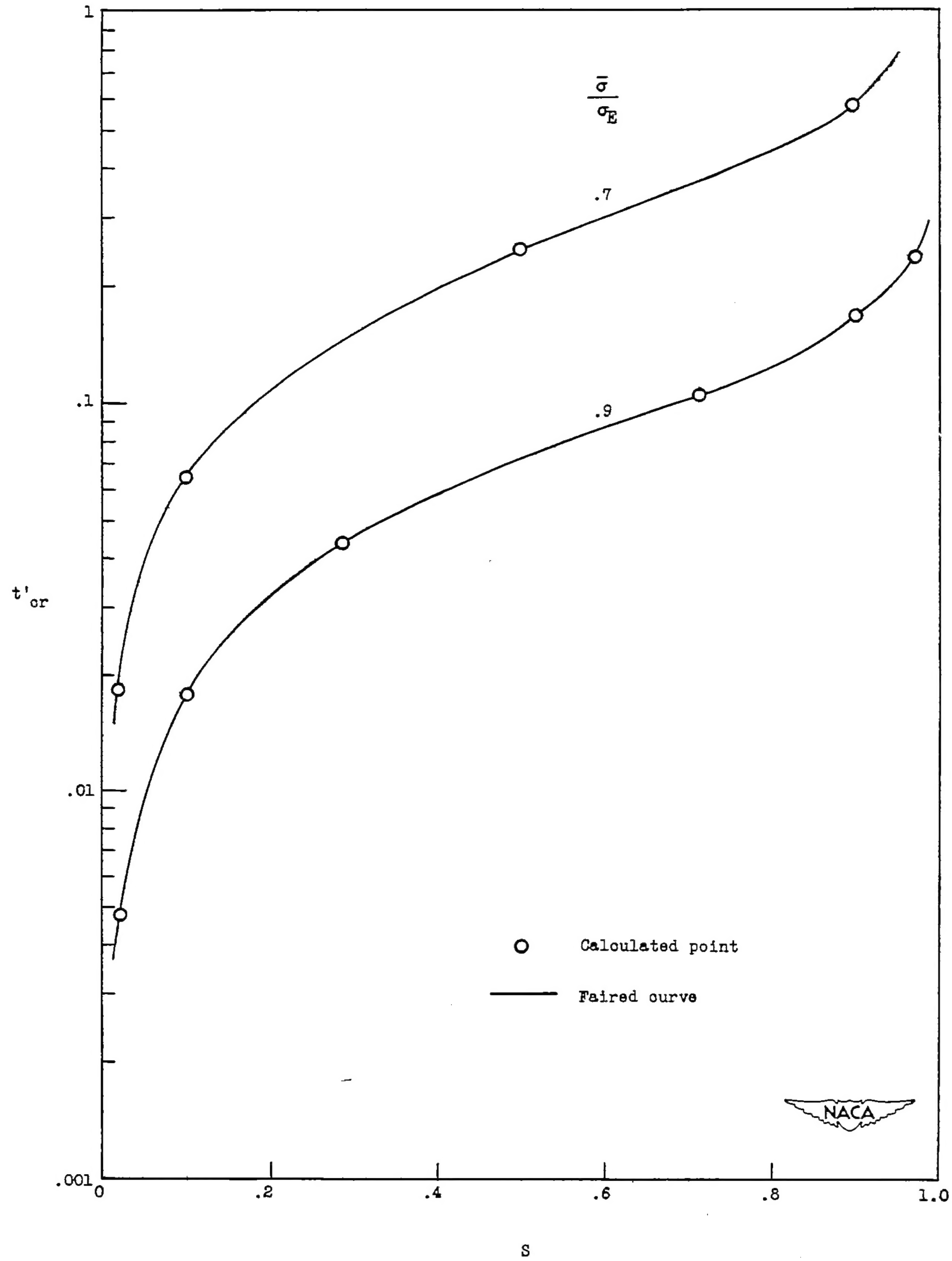

Figure 4.- Variation of lifetime parameter t'_{cr} with initial-straightness parameter S for two values of the load parameter $\bar{\sigma}/\sigma_E$ and for K = 1.

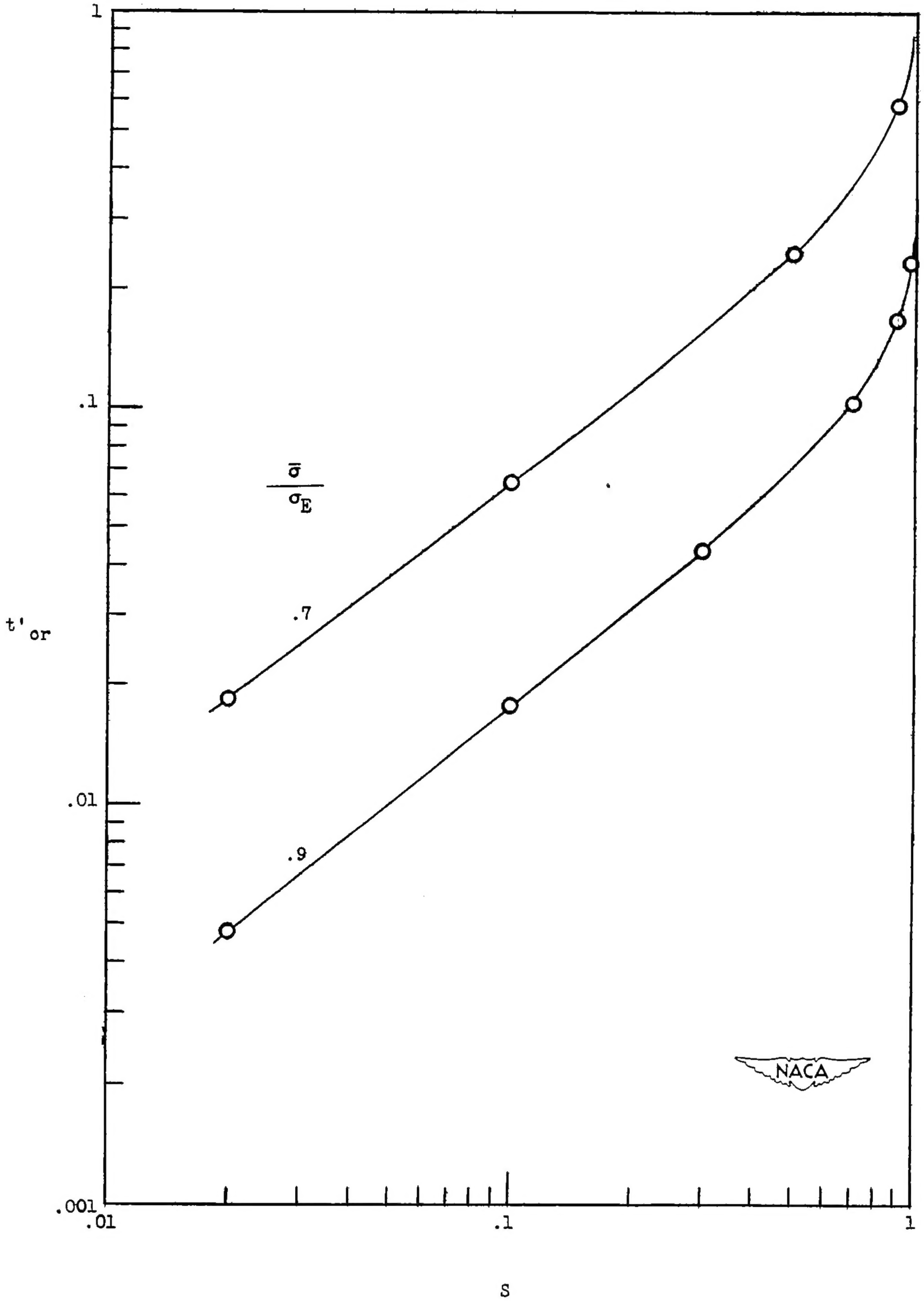

Figure 5.- Replot of figure 4 showing more clearly the nature of the curves in the region of $S = 0$.

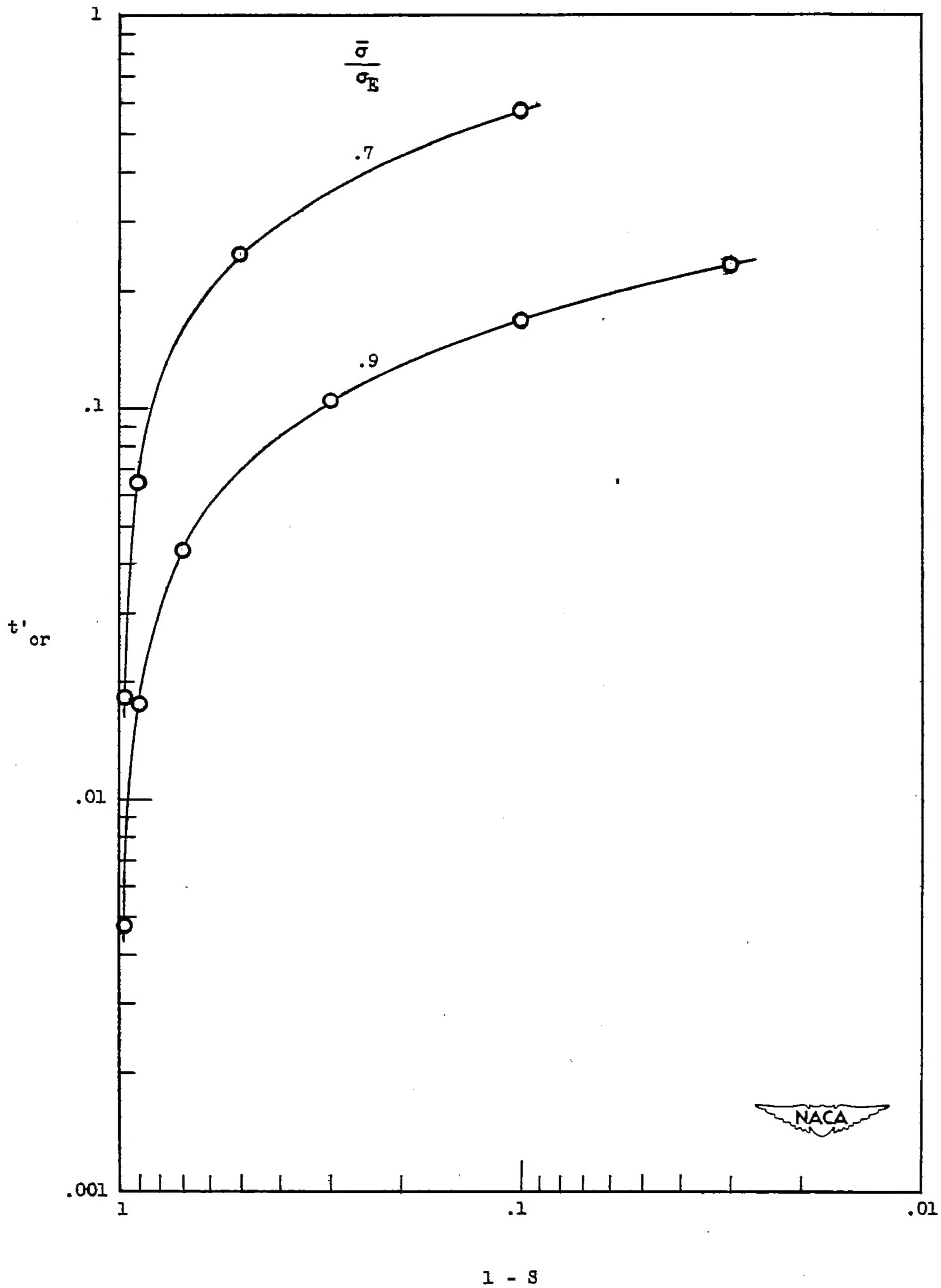

Figure 6.- Replot of figure 4 showing more clearly the nature of the curves in the region of S = 1.

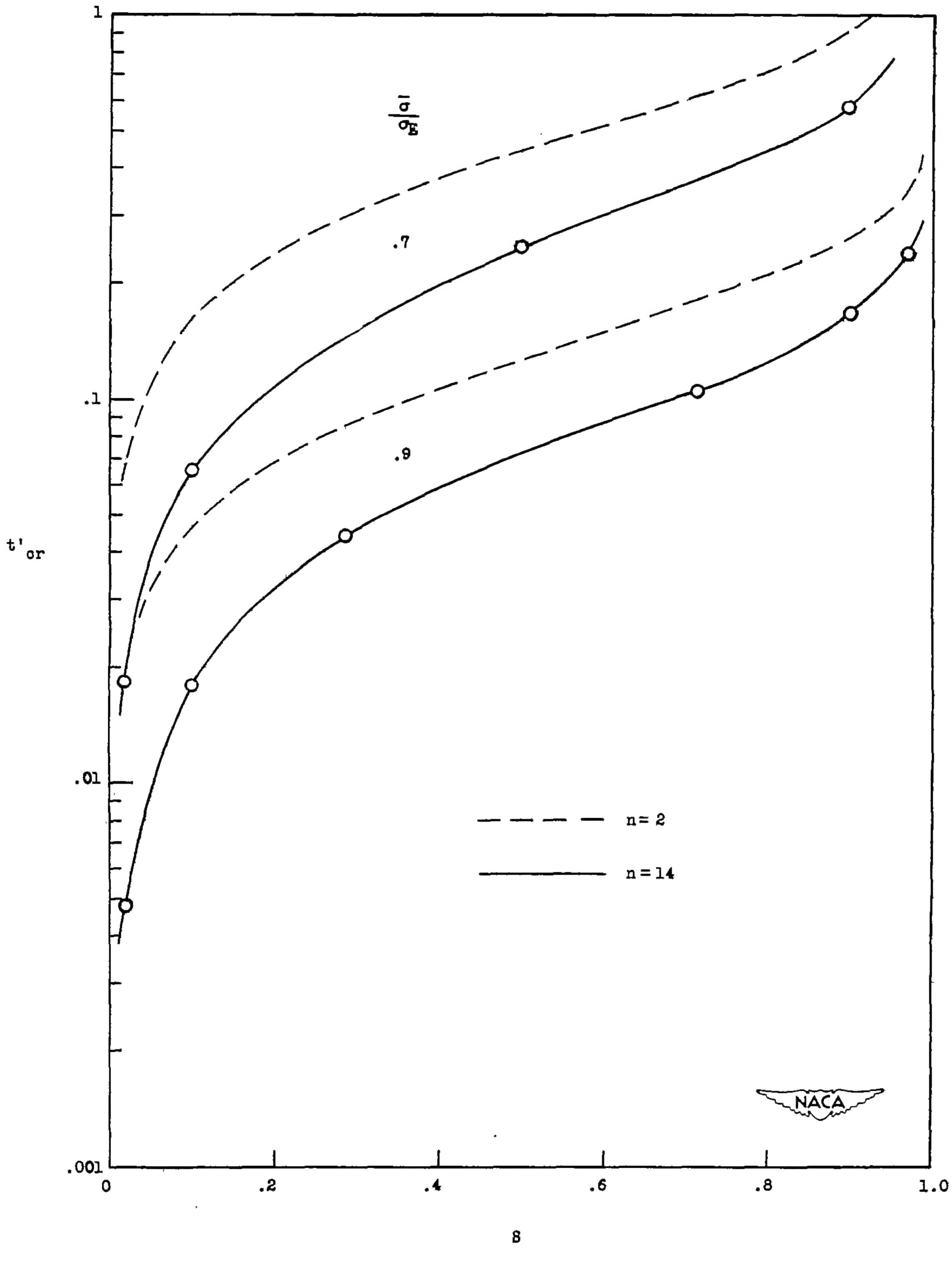

Figure 7.- Comparison of approximate solution, based on $n = 2$, with accurate solution, based on $n = 14$, for $K = 1$.